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On $4/n = 1/x + 1/y + 1/z$ and Iwaniec' Half Dimensional Sieve

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A well-known conjecture says that for any integer $n > 1$ the equation $4/n = 1/x + 1/y + 1/z$ has a solution in positive integers x, y, and z. By use of sieve methods we prove some asymptotic formulae and lower bounds for certain exceptional sets related to this problem. © 1994 Academic Press, Inc.

1. INTRODUCTION

In 1948, Erdös and Straus made the following

Conjecture. For any integer $n > 1$,

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \tag{1}$$

has a solution in positive integers x, y, z.

Although it has attracted a considerable amount of attention, the problem is still open. For the extensive literature concerning the conjecture as well as its generalizations by Sierpiński, Schinzel, and others, we refer to [3]. For results published after 1980, the reader may consult the updated list of references at the end of this paper.

In order to solve the original conjecture stated above, it obviously suffices to solve (1) for all primes q instead of n. Moreover, one can easily show that, if there is a solution of

$$\frac{4}{q} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z},$$

then either exactly one of the numbers x, y, z is divisible by q, or exactly two of them have a divisor q. In this paper we only consider the second case, namely the equation

$$\frac{4}{q} = \frac{1}{w} + \frac{1}{gq} + \frac{1}{hq} \tag{2}$$

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for a given prime q. The solvability of (2) is equivalent to the solvability of

$$(4g-1)(4h-1)=4qt+1, \qquad t \mid gh. \tag{3}$$

We are interested in the number of primes q, for which (3) has a solution, if t is fixed. Note that for $t=1$ the second condition in (3) is always satisfied.

It is known that for $n \not\equiv 1 \bmod 24$, solutions of (1) may be found constructively (with a little more effort the remaining residue class $n \equiv 1 \bmod 24$ can be reduced to an even thinner set of possible exceptions to the conjecture [11]. We take this into account by only considering primes $q \equiv l \bmod k$ in (3) for arbitrary positive integers k and l, $(k, l)=1$.

If w, g, h is a solution of (2) in positive integers, then $w(4gh-g-h)=ghq$. By [1, Satz 2], we have $q \nmid w$. Hence $q \mid (4gh-g-h)$, and gh/w is an integer. Thus we define

$$V(x; k, l; t)=\operatorname{card}\left\{q \leqslant x : q \equiv l \bmod k, \text{(2) unsolvable with } \frac{gh}{w}=t\right\}.$$

Furthermore, let

$$E(x, K)=\sum_{k<K} \max_{(l, k)=1}\left|\pi(x; k, l)-\frac{\operatorname{Li} x}{\varphi(k)}\right|.$$

THEOREM 1. *Let $p \not\equiv 3 \bmod 4$ or $p \nmid (4tl+1)$ for all primes $p \mid k$, and let $(k, l)=1$. Then*

$$V(x; k, l; t) \geqslant C_1(t, k)\frac{x}{(\log x)^{3/2}}$$

for a positive constant $C_1(t, k)$ only depending on t and k.

THEOREM 2. *Let $0<\varepsilon<\frac{1}{6}$. Let $p \not\equiv 3 \bmod 4$ or $p \nmid (4tl+1)$ for all $p \mid k$, and $(k, l)=1$. Then*

$$\text{(i)} \quad V(x; k, l; 1)=C_2(1, k)\frac{x}{(\log x)^{3/2}}(1+O(\varepsilon^{\gamma}))+\theta E(x, x^{1-\varepsilon}),$$

$$\text{(ii)} \quad V(x; k, l; t) \geqslant C_2(t, k)\frac{x}{(\log x)^{3/2}}(1+O(\varepsilon^{\gamma}))+\theta E(x, x^{1-\varepsilon}),$$

where

$$C_2(t, k)=\frac{\sqrt{2}}{\varphi(k)} \prod_{p \equiv 3 \bmod 4}\left(1-\frac{1}{p^2}\right)^{-1/2}\left(1-\frac{2}{p(p-2)}\right) \prod_{\substack{p \equiv 3 \bmod 4 \\ p \mid tk}} \frac{p-1}{p-2},$$

$$\gamma=\frac{1}{2}-\frac{1}{(2-\log 2)e},$$

and $|\theta| \leqslant 1$. The constant implied by $O(\ )$ only depends on t and k.

Halberstam's conjecture (see [10, Chap. 15]) would imply an unconditional version of Theorem 2. By more sophisticated methods, asymptotic formulae may be obtained not only for $t = 1$, but for all primes t (see [13]). It would be interesting to have analogues of Theorems 1 and 2, if t were allowed to assume any value from a given finite set of positive integers (see the final remarks).

I thank Professor E. Fouvry for his valuable indication of how to improve Theorem 1.

2. Preliminaries

Let $\mathscr{P}$ be a set of odd primes and $\mathscr{T} = \{t_1, ..., t_r\}$ be a set of positive integers.

Lemma 1. *Let d be squarefree with $p \in \mathscr{P}$ for all $p \mid d$. Let*

$$t_i \not\equiv t_j \quad \bmod p \tag{4}$$

for all $p \in \mathscr{P}$ and $i \neq j$. Then

$$\begin{aligned}\mathscr{B}_d(r) &:= \{1 \leqslant n \leqslant d : (4t_1 n + 1) \cdots (4t_r n + 1) \equiv 0 \bmod d\} \\ &= \bigcup_{d_1 \cdots d_r = d} \{1 \leqslant n \leqslant d : 4t_i n + 1 \equiv 0 \bmod d_i \ (1 \leqslant i \leqslant r)\} \qquad (\textit{disjoint}).\end{aligned}$$

Proof. Since d is squarefree, the union of sets above is obviously equal to the set $\mathscr{B}_d(r)$. It remains to show that any two sets in the union are disjoint. Suppose that n satisfies

$$\begin{aligned}&4t_1 n + 1 \equiv 0 \bmod d_1, ..., 4t_r n + 1 \equiv 0 \bmod d_r, \\ &4t_1 n + 1 \equiv 0 \bmod d'_1, ..., 4t_r n + 1 \equiv 0 \bmod d'_r\end{aligned}$$

with $d_1 \cdots d_r = d'_1 \cdots d'_r = d$. If there is a $p \mid d$ such that $p \mid d_i$ and $p \mid d'_j$ $(i \neq j)$, then $4t_i n + 1 \equiv 4t_j n + 1 \bmod p$, hence $p \mid n$ by (4). This yields the contradiction $4t_i n + 1 \not\equiv 0 \bmod p$. Therefore, $d_i = d'_i$ for all i.

Note that (4) is rather restrictive; it implies that $\operatorname{card} \mathscr{T} \leqslant \min_{p \in \mathscr{P}} p$.

Lemma 2. *Let d be squarefree with $p \in \mathscr{P}$ for all $p \mid d$. If (4) holds and*

$$p \nmid t_i \qquad (1 \leqslant i \leqslant r) \tag{5}$$

for all $p \in \mathscr{P}$, then

$$B_d(r) := \operatorname{card} \mathscr{B}_d(r) = \sum_{d_1 \cdots d_r = d} 1.$$

Particularly, $B_d(1)=1$, *and* $B_d(2)=\tau(d)$ *where* τ *denotes the divisor function.*

Proof. By Lemma 1 we have

$$B_d(r)=\sum_{d_1\cdots d_r=d}\operatorname{card}\{1\leqslant n\leqslant d: 4t_in+1\equiv 0 \bmod d_i\ (1\leqslant i\leqslant r)\}.$$

The congruence system

$$4t_1n+1\equiv 0 \bmod d_1, ..., 4t_rn+1\equiv 0 \bmod d_r$$

has exactly one solution mod $d_1\cdots d_r=d$, since each single congruence is solvable by (5). This proves the lemma.

3. The Half Dimensional Sieve

In this section our notation follows that used in [7]. Constants $c_1, c_2, ...$ may only depend on those parameters which are explicitly indicated. By p and q we always denote primes. For arbitrary positive integers t, k, l, $(k, l)=1$, and real $x, z>1$, let

$$\mathscr{A}=\mathscr{A}_{t,k}(x)=\{4tq+1 : q\leqslant x, q\equiv l \bmod k\},$$
$$\mathscr{P}=\{p\in\mathbb{P} : p\equiv 3 \bmod 4\},$$
$$P(z)=\prod_{\substack{p<z\\ p\in\mathscr{P}}} p.$$

For squarefree d satisfying $p\in\mathscr{P}$ for all $p\mid d$ let

$$\begin{aligned}\mathscr{A}_d&=\{a\in\mathscr{A} : a\equiv 0 \bmod d\}\\ &=\{4tq+1 : q\leqslant x, q\equiv l \bmod k, 4tq+1\equiv 0 \bmod d\}.\end{aligned}$$

Lemma 3. *There is an integer* l' *with* $(l', [d,k])=1$ *and*

$$|\mathscr{A}_d|=\begin{cases}\pi(x; [d,k], l') & \text{for } (t,d)=1 \text{ and } (d,k)\mid(4tl+1),\\ 0 & \text{otherwise.}\end{cases}$$

Proof. We consider the congruence system

$$m\equiv l \bmod k,\qquad 4tm+1\equiv 0 \bmod d \tag{6}$$

for the integer variable m. The second congruence of (6) is solvable iff $(t,d)=1$, and when Lemma 2 for $r=1$ is applied, there is a unique solution $l' \bmod [d, k]$ of (6) satisfying $(l', [d, k])=1$ iff $(d, k) \mid (4tl+1)$ (see [4, p. 21]). This proves the lemma.

Let

$$X=\frac{\operatorname{Li} x}{\varphi(k)}, \qquad \omega(p)=\begin{cases} p/\varphi(p) & \text{for } \ p \in \mathscr{P}, p \nmid tk, \\ 0 & \text{otherwise,} \end{cases} \tag{7}$$

where φ denotes Euler's function, and ω is multiplicative. Moreover, let

$$R(\mathscr{A}, d)=|\mathscr{A}_d|-\frac{\omega(d)}{d} X.$$

By Lemma 3, we have

LEMMA 4. *Let $p \not\equiv 3 \bmod 4$ or $p \nmid (4tl+1)$ for all $p \mid k$. Then*

$$R(\mathscr{A}, d)=\begin{cases} \pi(x; dk, l')-\operatorname{Li} x/\varphi(dk) & \textit{for} \quad (d, tk)=1 \textit{ and } p \in \mathscr{P}\ (p \mid d), \\ 0 & \textit{otherwise.} \end{cases}$$

Obviously, (Ω_1) in [7] is satisfied. It remains to check (Ω_2). By definition of $\omega(p)$, we have

$$\sum_{p \leqslant x} \frac{\omega(p)}{p} \log p=\sum_{\substack{p \leqslant x,\, p \in \mathscr{P} \\ p \nmid tk}} \frac{\log p}{p-1}=\sum_{\substack{p \leqslant x \\ p \in \mathscr{P}}} \frac{\log p}{p-1}+c_1(t, k).$$

Similarly

$$\sum_{p \leqslant x} \frac{\omega(p)}{p-\omega(p)} \log p=\sum_{\substack{p \leqslant x \\ p \in \mathscr{P}}} \frac{\log p}{p-2}+c_2(t, k).$$

Partial summation and Dirichlet's prime number theorem give for $j \in \{1, 2\}$, $x \geqslant 3$,

$$\sum_{\substack{p \leqslant x \\ p \in \mathscr{P}}} \frac{\log p}{p-j}=\frac{1}{2} \log x+c_3(j).$$

This yields (Ω_2) with constants K and L only depending on t and k.

The sieve method is used to estimate the sifting function

$$S(\mathscr{A}, z):=\operatorname{card}\{a \in \mathscr{A} : (a, P(z))=1\}.$$

PROPOSITION 1. *Let k satisfy the condition of Lemma 4. Then for $s \geqslant 1$ and sufficiently large z (depending on s), we have*

$$S(\mathscr{A}, z) \geqslant c_4(s; t, k) \operatorname{Li} x(\log z)^{-1/2}-E(x, z^s), \tag{8}$$

where c_4 is a positive constant.

Proof. Since (Ω_1) and (Ω_2) are satisfied, we obtain by Theorem 1 of [7] for $s \geqslant 1$ and sufficiently large z

$$S(\mathscr{A}, z) \geqslant c_5(s; t, k)\, \Omega(z)\, X - \sum_{\substack{d < z^s \\ d \mid P(z)}} |R(\mathscr{A}, d)|, \tag{9}$$

where

$$\Omega(z) := \prod_{p < z} \left(1 - \frac{\omega(p)}{p}\right) = \prod_{\substack{p < z,\, p \in \mathscr{P} \\ p \nmid tk}} \left(1 - \frac{1}{p-1}\right).$$

Applying partial summation and Dirichlet's prime number theorem, we get

$$-\log \Omega(z) = \sum_{\substack{p < z \\ p \in \mathscr{P}}} \frac{1}{p-1} + c_6(t, k) = \frac{1}{2} \log\log z + c_7(t, k).$$

Hence

$$\Omega(z) = c_8(t, k)(\log z)^{-1/2}$$

with a positive constant c_8. Combining this with (9) and (7) gives the first summand on the right hand side of (8). By Lemma 4, and summing $\sum R(\mathscr{A}, d)$ over all modules, we obtain the desired result.

We define

$$b^*(a) = \begin{cases} 1 & \text{if} \quad a = u^2 + v^2,\ (u, v) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

PROPOSITION 2. *Let k satisfy the condition of Lemma 4. Let $0 < \varepsilon < \frac{1}{6}$. Then*

$$\sum_{a \in \mathscr{A}} b^*(a) = C_2(t, k) \frac{x}{(\log x)^{3/2}} (1 + O(\varepsilon^{\gamma})) + \theta E(x, x^{1-\varepsilon}),$$

where $|\theta| \leqslant 1$, and the constant implied by $O(\ )$ only depends on t and k. C_1 and γ are defined as in Theorem 2.

Proof. By the definition of $\mathscr{A}$, we obviously have $A \leqslant 4tx + 1$ for $A := \max_{a \in \mathscr{A}} a$. On the other hand, by Bertrand's postulate in arithmetic progressions (see, for instance, [16]), we have

$$\pi(2y; k, l) - \pi(y; k, l) > 0$$

for $(k, l) = 1$ with k sufficiently large and $y \geqslant c_9(k)$ for a suitable constant c_9. This implies that there exists a positive constant $c_{10}(t, k)$, such that there is a prime $q \equiv l \bmod k$ satisfying

$$c_{10}(t, k)x < q \leqslant x$$

for $x \geqslant c_{11}(t, k)$. Hence $A = c_{12}(t, k)\, x$, where $c_{12} > 0$. Without loss of generality we may assume that $c_{12} \leqslant 1$. We choose $Q = (c_{12}x)^{\varepsilon}$. Then

$$(\log A)^{-1/2}\left(1 + O\left(e^{6K}\left(\frac{L + \log Q}{\log A}\right)^{\gamma}\right)\right) = (\log x)^{-1/2}\,(1 + O(\varepsilon^{\gamma})),$$

where the constant implied by $O(\ )$ only depends on t and k. By the assumption on c_{12}, we have $A/Q \leqslant x^{1-\varepsilon}$. When Lemma 4 is applied, and $\sum R(\mathscr{A}, d)$ is summed over all modules, Theorem 3 of [7] yields our theorem. (Note that the condition on $\mathscr{P}$ in Theorem 3 of [7] should read "$\mathscr{P} = \{p \equiv -1 \bmod 4\}$.")

4. Selberg's Upper Sieve

In this section we use a method due to Iwaniec [6]. This method enables us to avoid an "error term" $E(x, x^{1/2+\varepsilon})$ in Theorem 1 (the exponent $\frac{1}{2} + \varepsilon$ is just too big to be treated by Bombieri's theorem), which finally leads to the unconditional result.

For positive integers t and $a \equiv 3 \bmod 4$, $(a, t) = 1$, $N > 1$, let

$$\mathscr{B} = \mathscr{B}_{t,a}(N) = \left\{\frac{ap-1}{4t} : p \leqslant N, p \equiv 3 \bmod 4, ap \equiv 1 \bmod 4t\right\},$$

$$\mathscr{P} = \mathscr{P}_{t,a} = \{p \in \mathbb{P} : p \nmid 2ta\},$$

and $P(z)$ be as before. For squarefree d satisfying $p \in \mathscr{P}$ for all $p \mid d$, let

$$\mathscr{B}_d = \{b \in \mathscr{B} : b \equiv 0 \bmod d\}.$$

Lemma 5. *There is an integer l' with $(l', 4td) = 1$ and*

$$|\mathscr{B}_d| = \begin{cases} \pi(N; 4td, l') & \text{if } p \in \mathscr{P} \text{ for all } p \mid d, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. (See the proof of Lemma 3.)

Define

$$X = \frac{\operatorname{Li} N}{\varphi(4t)}, \qquad \omega(p) = \begin{cases} \dfrac{p}{\varphi(p)} & \text{for } p \in \mathscr{P}, \\ 0 & \text{otherwise,} \end{cases}$$

with $\omega(d)$ multiplicative. For

$$R(\mathscr{B}, d) = |\mathscr{B}_d| - \frac{\omega(d)}{d} X,$$

we have by Lemma 5

LEMMA 6. *For squarefree d,*

$$R(\mathscr{B}, d) = \begin{cases} \pi(N; 4td, l') - \dfrac{\operatorname{Li} N}{\varphi(4t)} & \text{if } p \in \mathscr{P} \text{ for all } p \mid d, \\ 0 & \text{otherwise,} \end{cases}$$

LEMMA 7. *For $a \equiv 3 \bmod 4$ and $(a, t) = 1$, we have*

$$\operatorname{card}\left\{p \leqslant N : p \equiv 3 \bmod 4, ap \equiv 1 \bmod 4t, \frac{ap-1}{4t} \in \mathbb{P}\right\}$$
$$\leqslant \frac{8}{\varphi(4t)} \prod_{p>2}\left(1 - \frac{1}{(p-1)^2}\right) \prod_{2<p \mid ta}\left(1 + \frac{1}{p-2}\right)$$
$$\times \frac{N}{(\log N)^2}\left(1 + O_t\left(\frac{\log\log N}{\log N}\right)\right).$$

Proof. By Theorem 3.10 of [4] with $K = 2ta$, we obtain for $S(\mathscr{B}, z)$, as defined in the preceding section,

$$S(\mathscr{B}, z) \leqslant 2 \prod_{p>2}\left(1 - \frac{1}{(p-1)^2}\right) \prod_{2<p \mid ta}\left(1 + \frac{1}{p-2}\right) \frac{X}{\log z}\left(1 + O\left(\frac{1}{\log z}\right)\right)$$
$$+ \sum_{\substack{d<z^2 \\ (d, 2ta)=1}} \mu^2(d)\, 3^{\nu(d)}\, |R(\mathscr{B}, d)|.$$

Using Lemma 6, and copying the proof of Theorem 3.12 in [4], we obtain the desired result. Note that we apply Bombieri's well-known theorem (see [4, Lemma 3.3], or [10, Chap. 15]) which states that, given any positive constant U, there is a positive constant V such that

$$E(x, x^{1/2}(\log x)^{-V}) \ll_U \frac{x}{(\log x)^U}. \tag{10}$$

LEMMA 8. *If $f(m) \geqslant 0$ is a multiplicative function satisfying*

$$f(p^j) \leqslant \frac{\gamma_1 \gamma_2^j}{p^j} \qquad (j \geqslant 2)$$

for some $\gamma_2 < 2$, *and*

$$\sum_{p \leqslant T} f(p) \log p \sim \tau \log T,$$

then

$$\sum_{m \leqslant T} f(m) \sim \frac{(\log T)^\tau}{\Gamma(1+\tau)} \prod_{p \leqslant T} (1 + f(p) + f(p^2) + \cdots)\left(1 - \frac{1}{p}\right)^\tau.$$

Proof. This is a version of Wirsing's theorem on sums of multiplicative functions (see [19] or [14, III, 6]).

PROPOSITION 3. *Let* $\frac{1}{3} < r < \frac{1}{2}$, $z = x^r$. *Then for some positive constant* $c_{13}(t)$,

$$\begin{aligned} &\operatorname{card}\{q \leqslant x : 4tq + 1 = p_1 p_2 m \text{ for some } p_1, p_2 \in \mathscr{P}_3, p_1, p_2 \geqslant z, m \in \mathscr{M}\} \\ &\qquad \leqslant c_{13}(t) \frac{1}{r^2} (1 - 2r)^{1/2} \log \frac{1-r}{r} \frac{x}{(\log x)^{3/2}}, \end{aligned} \tag{11}$$

where $\mathscr{M}$ *denotes the set of positive integers only having prime factors* $\equiv 1 \bmod 4$, *and* $\mathscr{P}_3$ *the set of primes* $\equiv 3 \bmod 4$.

Proof. Let $D = D_t(x, z)$ be the cardinality defined in (11). Let $Y = 4tx + 1$. Then we have

$$D = \sum_{\substack{m \in \mathscr{M} \\ m \leqslant Y/z^2}} \sum_{\substack{p_1 \in \mathscr{P}_3,\, (p_1 m, t) = 1 \\ z \leqslant p_1 \leqslant Y/zm}} \sum_{\substack{p_2 \in \mathscr{P}_3,\, z \leqslant p_2 \leqslant Y/p_1 m \\ p_1 p_2 m \equiv 1 \bmod 4t \\ (p_1 p_2 m - 1)/4t \in \mathbb{P}}} 1.$$

Applying Lemma 7 to the innermost sum with $p = p_2$, $N = Y/p_1 m$, and $a = p_1 m$, we obtain

$$\begin{aligned} D &\leqslant \frac{8}{\varphi(4t)} \prod_{p > 2} \left(1 - \frac{1}{(p-1)^2}\right) \frac{Y}{(\log Y)^2} \\ &\quad \times \sum_m \sum_{p_1} \frac{1}{p_1 m} \prod_{2 < p \mid tp_1 m} \left(1 + \frac{1}{p-2}\right)\left(\frac{\log Y}{\log(Y/p_1 m)}\right)^2 \left(1 + O_t\left(\frac{\log \log N}{\log N}\right)\right) \\ &\leqslant c_{14}(t) \frac{x}{(\log x)^2} \sum_{\substack{m \in \mathscr{M} \\ m \leqslant Y/z^2}} \sum_{\substack{p_1 \in \mathscr{P}_3 \\ z \leqslant p_1 \leqslant Y/zm}} \frac{1}{p_1 m} \\ &\quad \times \prod_{p \mid m} \left(1 + \frac{1}{p-2}\right)\left(\frac{\log Y}{\log(Y/p_1 m)}\right)^2 (1 + o_t(1)), \end{aligned}$$

where

$$c_{14}(t)=\frac{16(4t+1)}{\varphi(4t)}\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)\prod_{2<p\mid t}\left(1+\frac{1}{p-2}\right).$$

Clearly, in the inner sum we have

$$\frac{\log Y}{\log(Y/p_1 m)}\leqslant\frac{\log Y}{\log z}=\frac{\log(4tx+1)}{\log x^r}=\frac{1}{r}(1+o_t(1)).$$

Thus

$$D\leqslant\frac{c_{14}(t)}{r^2}\frac{x}{(\log x)^2}\sum_{\substack{m\in\mathcal{M}\\ m\leqslant Y/z^2}}\frac{1}{m}\prod_{p\mid m}\left(1+\frac{1}{p-2}\right)\sum_{\substack{p_1\in\mathcal{P}_3\\ z\leqslant p_1\leqslant Y/zm}}\frac{1}{p_1}(1+o_t(1)).$$

By Dirichlet's prime number theorem

$$\begin{aligned}\sum_{\substack{p_1\in\mathcal{P}_3\\ z\leqslant p_1\leqslant Y/zm}}\frac{1}{p_1}&\leqslant\frac{1}{2}\left(\log\log\frac{4tx+1}{z}-\log\log z\right)(1+o(1))\\&=\frac{1}{2}\log\left(\frac{\log(4tx+1)/z}{\log z}\right)(1+o(1))\\&=\frac{1}{2}\log\frac{1-r}{r}(1+o(1)).\end{aligned}$$

Therefore

$$D\leqslant c_{14}(t)\frac{1}{r^2}\log\frac{1-r}{r}\frac{x}{(\log x)^2}\sum_{\substack{m\in\mathcal{M}\\ m\leqslant Y/z^2}}\frac{1}{m}\prod_{p\mid m}\left(1+\frac{1}{p-2}\right)(1+o_t(1)).$$

The function

$$f(m)=\begin{cases}\dfrac{1}{m}\displaystyle\prod_{p\mid m}\left(1+\frac{1}{p-2}\right) & \text{for } m\in\mathcal{M},\\ 0 & \text{otherwise}\end{cases}$$

is multiplicative, and

$$f(p^j)=\begin{cases}p^{-j}\left(1+\dfrac{1}{p-2}\right) & \text{for } p\equiv 1 \bmod 4,\\ 0 & \text{otherwise.}\end{cases}$$

Hence

$$\prod_{p \leqslant T} (1 + f(p) + f(p^2) + \cdots)\left(1 - \frac{1}{p}\right)^{1/2}$$

$$= \prod_{\substack{p \leqslant T \\ p \equiv 1 \bmod 4}} \left(1 + \frac{1}{p}\left(1 + \frac{1}{p-2}\right) + \frac{1}{p^2}\left(1 + \frac{1}{p-2}\right) + \cdots\right)\left(1 - \frac{1}{p}\right)^{1/2}$$

$$\times \prod_{\substack{p \leqslant T \\ p \not\equiv 1 \bmod 4}} \left(1 - \frac{1}{p}\right)^{1/2}$$

$$= \prod_{\substack{p \leqslant T \\ p \equiv 1 \bmod 4}} \left(1 + \frac{1}{p(p-2)}\right)\left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots\right)\left(1 - \frac{1}{p}\right)^{1/2}$$

$$\times \prod_{\substack{p \leqslant T \\ p \not\equiv 1 \bmod 4}} \left(1 - \frac{1}{p}\right)^{1/2}$$

$$\leqslant \prod_{p \equiv 1 \bmod 4} \left(1 + \frac{1}{p(p-2)}\right) + \prod_{\substack{p \leqslant T \\ p \equiv 1 \bmod 4}} \left(1 - \frac{1}{p}\right)^{-1/2} \prod_{\substack{p \leqslant T \\ p \not\equiv 1 \bmod 4}} \left(1 - \frac{1}{p}\right)^{1/2}$$

$$\leqslant 2 \prod_{\substack{p \leqslant T \\ p \equiv 1 \bmod 4}} \left(1 - \frac{1}{p}\right)^{-1/2} \prod_{\substack{p \leqslant T \\ p \not\equiv 1 \bmod 4}} \left(1 + \frac{1}{p}\right)^{-1/2}$$

$$= 2 \prod_{p \leqslant T} \left(1 - \frac{1}{p}\chi_4(p)\right)^{-1/2} = 2\left(\frac{\pi}{4}\right)^{1/2} (1 + o(1)) = \sqrt{\pi}\,(1 + o(1)).$$

Applying Lemma 8 with $T = Y/z^2$, we obtain

$$D \leqslant c_{14}(t) \frac{\sqrt{\pi}}{\Gamma(\frac{3}{2})} \frac{x}{(\log x)^2} \left(\log \frac{4tx+1}{z^2}\right)^{1/2} \frac{1}{r^2} \log \frac{1-r}{r} (1 + o_t(1))$$

$$\leqslant c_{14}(t) \frac{1}{r^2} \sqrt{1 - 2r} \log \frac{1-r}{r} \frac{x}{(\log x)^{3/2}} (1 + o_t(1)),$$

which proves the proposition.

5. Proof of the Theorems

It is easily seen that the positive integers w, g, h solve (2), if and only if there is a positive integer t such that

$$(4g-1)(4h-1) = 4tq + 1, \qquad t \mid gh. \tag{12}$$

Thus we have

$$\begin{aligned} V(x;k,l;t) &= \operatorname{card}\{q \leqslant x : q \equiv l \bmod k,\ (12)\ \text{unsolvable}\} \\ &\geqslant \operatorname{card}\{q \leqslant x : q \equiv l \bmod k,\ (4g-1)(4h-1) = 4tq+1\ \text{unsolvable}\} \\ &= S(\mathscr{A}, \sqrt{4tx+2}) \end{aligned}$$

with the terminology of Section 3.

Clearly, for $\mathscr{P} = \{p \in \mathbb{P} : p \equiv 3 \bmod 4\}$

$$\begin{aligned} &S(\mathscr{A}, z) - S(\mathscr{A}, \sqrt{4tx+1}) \\ &\quad = \operatorname{card}\{a \in \mathscr{A} : p_1 \mid a \text{ for some } p_1 \in \mathscr{P}, p_1 \geqslant z; \\ &\qquad\qquad p_2 \geqslant z \text{ for all } p_2 \in \mathscr{P}, p_2 \mid a\} \\ &\quad = \operatorname{card}\{q \leqslant x : q \equiv l \bmod k, p_1 \mid (4tq+1) \text{ for some } p_1 \in \mathscr{P}, p_1 \geqslant z; \\ &\qquad\qquad p_2 \geqslant z \text{ for all } p_2 \in \mathscr{P}, p_2 \mid (4tq+1)\} \\ &\quad = \operatorname{card}\{q \leqslant x : q \equiv l \bmod k, 4tq+1 = p_1 p_2 m \text{ for some} \\ &\qquad\qquad z \leqslant p_1, p_2 \in \mathscr{P}, m \in \mathscr{M}\} \\ &\quad \leqslant \operatorname{card}\{q \leqslant x : 4tq+1 = p_1 p_2 m \text{ for some } z \leqslant p_1, p_2 \in \mathscr{P}, m \in \mathscr{M}\}. \end{aligned}$$

Now let

$$s = 1, \qquad r = \tfrac{1}{2} - \delta\ (\delta > 0), \qquad z = x^r.$$

Then by Propositions 1 and 3

$$S(\mathscr{A}, \sqrt{4tx+1}) \geqslant c_{15}(t,k) \frac{x}{(\log x)^{3/2}} - E(x, x^{1/2-\delta}) - c_{16}(t,\delta) \frac{x}{(\log x)^{3/2}},$$

where $c_{15}(t,k) > 0$, and $c_{16}(t,\delta) \to 0$ for $\delta \to 0$. Thus

$$S(\mathscr{A}, \sqrt{4tx+1}) \geqslant C_1(t,k) \frac{x}{(\log x)^{3/2}}$$

for some positive constant C_1 by Bombieri's theorem (see Eq. (10)), which proves Theorem 1.

In order to prove Theorem 2, we first state the subsequent well-known fact.

LEMMA 9. *Let a be an odd integer. Then there are integers u and v, $(u, v) = 1$, satisfying $a = u^2 + v^2$ if and only if $p \nmid a$ for all primes* $p \equiv 3 \bmod 4$.

Proof. See [8, Satz 164].

By (13) and Lemma 9, we have

$$\begin{aligned} V(x; k, l; t) &\geqslant \operatorname{card}\{q \leqslant x : q \equiv l \bmod k, p \mid (4tq+1) \Rightarrow p \equiv 1 \bmod 4\} \\ &= \operatorname{card}\{q \leqslant x : q \equiv l \bmod k, 4tq+1 = u^2+v^2, (u, v) = 1\} \\ &= \sum_{a \in \mathscr{A}} b^*(a), \end{aligned}$$

where equality obviously holds for $t = 1$. By Proposition 2, this proves Theorem 2.

6. Final Remarks

If we allow in our theorems the two values t_1 and t_2 instead of the fixed value t, then Lemma 2 would be applicable with $r = 2$. This leads to a one dimensional sieving problem. Respectively, fixed values $t_1, ..., t_r$ result in higher dimensional sieves.

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